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Frobenius Differential-Algebraic Universums on Complex Algebraic Curves

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"Popular series is bound to express popular views."
Oscar Wilde

Abstract—In terms of differential generators and differential relations for a finitely generated commutative-associative differential C-algebra A (with a unit element) we study and determine necessary and sufficient conditions for the fact that under any Taylor homomorphism $\widetilde{\psi}_M : A \to \mathbb{C}[[z]]$ the transcendence degree of the image $\widetilde{\psi}_M(A)$ over C does not exceed 1 ($\widetilde{\psi}_M(a) \overset{\text{def}}{=} \sum\limits_{m=0}^{\infty} \psi_M(a^{(m)})\frac{z^m}{m!}$, where $a \in A$, $M \in \operatorname{Spec}_{\mathbb{C}} A$ is a maximal ideal in A, $a^{(m)}$ is the result of m-fold application of the signature derivation of the element a, and ψ_M is the canonic epimorphism $A \to A/M$).

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1. Introduction. These notes are not written for "proud scientists" (see [1, Introduction]), but for those who following us will want to read them.

1.1. Differential C-algebras of Gerasimova. Let $F_2 \overset{\text{def}}{=} C[x^{(i)}, y^{(j)} | i, j = 0, 1, 2, \dots]$ be a free differential C-algebra with two free generators $x \overset{\text{def}}{=} x^{(0)}$, $y \overset{\text{def}}{=} y^{(0)}$, and the signature differentiation in it maps $x^{(i)}$, $y^{(j)}$ to $x^{(i+1)}$, $y^{(j+1)}$, respectively. Order all the monomials of degree $\leq m$ of x and y in the following way:

x^m, $x^{m-1}y$, $x^{m-2}y^2$, $\dots$, xy^{m-1}, y^m; x^{m-1}, $x^{m-2}y$, $\dots$, xy^{m-2}, y^{m-1}; $\dots$; x^2, xy, y^2; x, y; 1.

Denote the Wronskian of these elements (this is a particular differential polynomial from F_2) by $H_m(x, y)$.

Specify the differential C-algebra $G_{m,n}$ by the differential generators $g_1, \dots, g_n$ and the differential determining relations of two following types:

(a) $H_m(g_i, g_i') = 0$ $(i = 1, 2, \dots, n)$;

(b) $H_m(g_i, g_j) = 0$ $(i \geq j; i, j = 1, 2, \dots, n)$.

Proposition 1. *For any maximal ideal M of the differential C-algebra $G_{m,n}$ the image of this algebra under the Tailor homomorphism $\widetilde{\psi}_M : G_{m,n} \to \mathbb{C}[[z]]$ possesses the following properties:*

(i) the transcendence degree of $\widetilde{\psi}_M(G_{m,n})$ does not exceed one;

(ii) the C-algebra $\widetilde{\psi}_M(G_{m,n})$ is finitely generated as a commutative-associative algebra;

(iii) all power series $\bar{g}_1(z) \overset{\text{def}}{=} \widetilde{\psi}_M(g_1)$, $\dots$, $\bar{g}_n(z) \overset{\text{def}}{=} \widetilde{\psi}_M(g_n)$ converge in a certain neighborhood of zero in the complex plane C^1.

Proof. Since a Tailor homomorphism is a differential homomorphism, then $\widetilde{\psi}_M(G_{m,n})$ is a differential C-subalgebra in $\mathbb{C}[[z]]$ relative to the differentiation $\frac{d}{dz}$ and $\bar{g}_1(z), \dots, \bar{g}_n(z)$ are differential generators of $\widetilde{\psi}_M(G_{m,n})$. If all the power series $\bar{g}_i(z)$ $(i = 1, \dots, n)$ are constants, then the assertion of the proposition is trivial.

Let $\bar{g}_q(z) \neq \text{const}$ for some $q \in \{1, \dots, n\}$. It is known (see, e.g., [2]) that if the Wronskian $|f_1(t), \dots, f_s(t)|$ of the power series $f_1(t), \dots, f_s(t) \in k[[t]]$ equals zero, then these power series are linearly dependent over k. Therefore, the definition of $H_m(x, y)$ and the equalities $H_m(g_q, g_q') = 0$, $0 = \widetilde{\psi}_M(H_m(g_q, g_q')) = H_m(\widetilde{\psi}_M(g_q), \widetilde{\psi}_M(g_q')) = H_m(\bar{g}_q, \bar{g}_q')$ imply that the power series $\bar{g}_q(z)$, $\bar{g}_q'(z)$ must be connected by a certain nontrivial polynomial relation $H(\bar{g}_q, \bar{g}_q') = 0$, where $H(u, v) \in C[u, v]$ and the degree of H does not exceed m. Take the polynomial of the least degree from those polynomials. In this case we have

a) $H(u, v)$ is an irreducible polynomial in $C[u, v]$ (because $C[[z]]$ does not contain divisors of zero);

b) $\frac{\partial H}{\partial v} \neq 0$ (otherwise $H(u, v) = H(u, 0) = \alpha \cdot u + \beta \cdot 1$ and $\bar{g}_q(z) = \text{const}$);

c) $\frac{\partial H}{\partial v}|_{u=\bar{g}_q, v=\bar{g}_q'} \neq 0$ (since the degree of $\frac{\partial H}{\partial v}$ is strictly less than the degree of H),

and we get the thoroughly studied situation (see [3]).

Theorem (O.V. Gerasimova). *There exists a unique up to an isomorphism differential domain of integrity W_H generated by one differential generator w such that $w \neq \text{const}$ and $H(w, w') = 0$, where $H(u, v)$ is an irreducible polynomial in $C[u, v]$. Moreover, the transcendence degree of W_H over C is equal to one and the C-algebra W_H is finitely generated as a commutatively associative algebra.*

Therefore, the differential C-subalgebra A_q generated by $\bar{g}_q(z)$ in $C[[z]]$ and its field of quotients $Q(A_q)$ have the transcendence degree over C equal to one.

Similarly, using determining relation of type (b) of the algebra $G_{m,n}$ and the Wronskian relations $H_m(\bar{g}_i, \bar{g}_j) = \widetilde{\psi}_M(H_m(g_i, g_j)) = \widetilde{\psi}_M(0) = 0$, we conclude that the elements $\bar{g}_j$ are algebraic over the field $Q(A_q)$, i.e., the field of quotients of the C-algebra $\widetilde{\psi}_M(G_{m,n})$ coincides with $Q(A_q)[\bar{g}_1, \dots, \bar{g}_n]$ and is a finite extension of the field $Q(A_q)$. Therefore, $\deg_C \widetilde{\psi}_M(G_{m,n}) = 1$, which proves property (i) of the C-algebra $\widetilde{\psi}_M(G_{m,n})$.

As was shown in [3] (see Theorem 1 and its corollary), properties (ii) and (iii) of the C-algebra $\widetilde{\psi}_M(G_{m,n})$ follow from property (i). Proposition 1 is completely proved.

1.2. Notations and terminology. Below we use the following notations and definitions for any finitely generated differential C-algebra B (with a unit):

$\deg_C B$ is the transcendence degree of B over C;

$\mathrm{Spec}_C B$ is the set of all maximal ideals in B;

$\widetilde{\psi}_M : B \to C[[z]]$ is the Tailor homomorphism at the point $M \in \mathrm{Spec}_C B$;

$\mathrm{RANG}_C B \stackrel{\text{def}}{=} \max_{M \in \mathrm{Spec}_C B} \deg_C \widetilde{\psi}_M(B)$ is the rank of the differential C-algebra B;

$\widetilde{\psi}_{M_0}(B)$ is the *germ of trajectory at the point* $M_0 \in \mathrm{Spec}_C B$;

$\{M \in \mathrm{Spec}_C B | M \supseteq \mathrm{Ker}\, \widetilde{\psi}_{M_0}\}$ is the *closure of the orbit* passing through the point $M_0 \in \mathrm{Spec}_C B$;

$\{M \in \mathrm{Spec}_C B|$ all the series from $\widetilde{\psi}_M(B)$ converge in a certain neighborhood of zero$\}$ is the *analytic spectrum* of the differential C-algebra B.

This terminology allows us to reformulate Proposition 1 and main results of the authors [3] (see Theorem 1 and its corollary) in the following way.

Proposition 2. *If the rank of a finitely generated differential C-algebra B is equal to one, then its spectrum is analytic and the closure of the orbit passing through an arbitrary point of the spectrum is an irreducible affine algebraic curve* (*if not this point itself*).

Obviously, the class of finitely generated differential C-algebras of rank ≤ 1 is closed relative to the operation of determination of homomorphic images and localizations of such algebras with respect to a non-nilpotent element. According to Proposition 1, it contains all algebras $G_{m,n}$. Are there any significantly different objects in the selected class? The answer will be given below.

And now we offer the reader to answer two questions (as an exercise).

Question 1. What are countable differential C-algebras (with the unit and without nil-elements) having zero rank?

Question 2. What is the rank of the free differential C-algebra $F_1 \stackrel{\text{def}}{=} C[x^{(0)}, x^{(1)}, x^{(2)}, \dots]$ with one free generator $x \stackrel{\text{def}}{=} x^{(0)}$ $((x^{(i)})' \stackrel{\text{def}}{=} x^{(i+1)})$?

2. Affine differential C-algebras. Let X be an affine complex algebraic variety, $C[X]$ be its algebra of regular functions ($X = \mathrm{Spec}_C\, C[X]$).

Lemma 1. *If the variety X is irreducible and is not a point, then it cannot be a union of a countable number of its proper subvarieties.*

Proof (E. B. Vinberg). Induction over the dimension of the variety X.

A differential C-algebra $C[X]$ relative to some fixed (nonzero) differentiation D is said to be an affine differential C-algebra; we denote it by $C_D[X]$. (The differentiation D specifies a vector field on the variety X and the notions defined above, namely, the germ of trajectory at a point $M \in X$, the closure of an orbit passing through M, the Cauchy problem admit natural interpretation.)

The following proposition describes affine differential C-algebras of rank 1.

Proposition 3. *If $a_1, \dots, a_q$ are arbitrary differential generators of an affine differential algebra $C_D[X]$ of an irreducible variety X, then $\mathrm{RANG}_C\, C_D[X] \leq 1$ if and only if $C_D[X]$ is a* (*differential*) *homomorphic image of the C-algebra $G_{m,q}$ $(g_1 \to a_1, \dots, g_q \to a_q)$ for some natural m, i.e., the following differential relations hold in $C_D[X]$:*

(a) $H_m(a_i, a_i') = 0$ $(i = 1, 2, \dots, q)$;

(b) $H_m(a_i, a_j) = 0$ $(i \geq j; i = 1, 2, \dots, q)$.

Proof. Let I_m be a differential ideal in $C_D[X]$ corresponding to differential relations (a), (b). This ideal defines a certain affine subvariety X_m in X. If $\mathrm{RANG}_C C_D[X] \leq 1$, then for an arbitrary maximal ideal

$M \in X$ in the image of $\widetilde{\psi}_M(C_D[X])$ the elements $\bar{a}_i \stackrel{\text{def}}{=} \widetilde{\psi}_M(a_i)$, $\bar{a}_i' \stackrel{\text{def}}{=} \widetilde{\psi}_M(D(a_i))$ $(i = 1, \dots, q)$ must be linked (pairwise) by some nontrivial polynomial relations. By n we denote the maximal degree of such polynomials, but in this case all Wronskians $H_n(\bar{a}_i, \bar{a}_i') = \widetilde{\psi}_M(H_n(a_i, a_i'))$, $H_n(\bar{a}_i, \bar{a}_j) = \widetilde{\psi}_M(H_n(a_i, a_j))$ are equal to zero in power series of $C[[z]]$. Therefore, $M \in X_n$ and $X = \bigcup_{n=0}^{\infty} X_n$, i.e., the irreducible variety X is a countable union of its subvarieties. Lemma 1 implies that $X = X_m$ for some m, which proves the proposition.

Corollary. *In an arbitrary affine differential algebra $C_D[X]$ of rank 1 (X is an irreducible variety) any (differential) finitely generated C-subalgebra A relative to the differentiation D is a homomorphic image of some C-algebra $G_{m,s}$, in particular, its rank does not exceed one.*

Proof. Supplement the differential generators $a_1, \dots, a_s$ of the C-subalgebra A up to a finite system of differential generators $C_D[X]$. According to the facts proved above, all relations (a), (b) from Proposition 3 must hold on this system of generators for some natural number m. But these relations also hold for the subset of differential generators $a_1, \dots, a_s$, i.e., A is a homomorphic image of $G_{m,s}$ $(g_i \to a_i)$. The corollary is completely proved.

It seems to us that here we could make a small respite in our presentation.

3. Coulomb field and its rank. Define a commutatively associative differential C-algebra R (with the unit) by the differential generators x, y, r, r^{-1} and the differential determining relations

$$x'' = -\frac{4\pi^2 k}{r^3} \cdot x, \; y'' = -\frac{4\pi^2 k}{r^3} \cdot y, \; r^2 = x^2 + y^2, \; r \cdot r^{-1} = 1,$$

where $0 \neq k \in C$. By $R_{\sigma_{0,1}}$ we denote the localization of the algebra R with respect to the element $\sigma_{0,1} \stackrel{\text{def}}{=} x \cdot y' - x' \cdot y$. As was shown in [3–5], for an arbitrary maximal ideal $M \in \operatorname{Spec}_C R_{\sigma_{0,1}}$ the power series $\bar{x}(z) \stackrel{\text{def}}{=} \widetilde{\psi}_M(x)$, $\bar{y}(z) \stackrel{\text{def}}{=} \widetilde{\psi}_M(y)$ are related as

$$H(\bar{x}, \bar{y}) = 0, \quad 0 \neq \bar{x} \cdot \bar{y}' - \bar{x}' \cdot \bar{y} = \sigma = \bar{x}(0) \cdot \bar{y}'(0) - \bar{x}'(0) \cdot \bar{y}(0) \in C,$$

where $H(u, v)$ is an irreducible quadratic polynomial. These relations (due to the absence of singular points on the curve $H(u, v) = 0$) successively imply the following properties:

(i) $L(\bar{x}, \bar{y}) \cdot (\bar{x}(z), \bar{y}(z))' = \sigma \cdot (-\frac{\partial H}{\partial v}, \frac{\partial H}{\partial u})|_{u=\bar{x}, v=\bar{y}}$, where $L(u, v) \stackrel{\text{def}}{=} \frac{\partial H}{\partial u} \cdot u + \frac{\partial H}{\partial v} \cdot v$;

(ii) $L(\bar{x}, \bar{y})$ is an invertible element in $\widetilde{\psi}_M(R_{\sigma_{0,1}})$, which coincides with $\bar{r}(z) \stackrel{\text{def}}{=} \widetilde{\psi}_M(r)$ up to a numeric multiplier (see formulas of Nikchemny in [4]);

(iii) the C-algebra $\widetilde{\psi}_M(R_{\sigma_{0,1}})$ is generated by the three elements $\bar{x}$, $\bar{y}$, $(L(\bar{x}, \bar{y}))^{-1}$ as a commutatively associative algebra.

Therefore, the field of quotients of the C-algebra $\widetilde{\psi}_M(R_{\sigma_{0,1}})$ coincides with the field of rational functions $C(X_H)$ of the quadratic curve X_H (given by the equation $H(u, v) = 0$) which (see [6]) in its turn is a field of complex rational functions of one variable. Therefore, the degree of transcendence of $\widetilde{\psi}_M(R_{\sigma_{0,1}})$ is equal to 1 for any $M \in \operatorname{Spec}_C R_{\sigma_{0,1}}$ and $\operatorname{RANG}_C R_{\sigma_{0,1}} = 1$ in the problem of two bodies[1].

Since $R_{\sigma_{0,1}}$ is an affine differential C-algebra and the domain of integrity, the corollary from Proposition 3 implies that any one-generated differential C-subalgebra in $R_{\sigma_{0,1}}$ has rank ≤ 1 and its differential generator has to satisfy one of the characteristic relations $H_m(w, w') = 0$ in it. What is the dependence of the number m on the choice of $w \in R_{\sigma_{0,1}}$?

If $w = r$, then the answer immediately follows from the “fundamental” equality

$$\frac{1}{2} \cdot ((\bar{r}'(z))^2 + \frac{\sigma^2}{\bar{r}^2(z)}) - \frac{4 \cdot \pi^2 \cdot k}{\bar{r}(z)} = E/m_e(E, \sigma \in C)$$

stating that the differential monomials $r^2 \cdot (r')^2$, r^2, r, 1 become linearly dependent after application of any Tailor homomorphism to them. Therefore, $m = 4$. Clearly, the relation $H_4(r, r') = 0$ very roughly reflects the specific situation $w = r$ because we can replace $H_4(r, r')$ by the Wronskian $|r^2 \cdot (r')^2, r^2, r, 1|$ in it.

Question. Why since the time of Robert Hooke’s predecessors there is nothing like this for $w = \alpha \cdot x + \beta \cdot y$ $(\alpha, \beta \in C)$ in tutorials?[2]

[1] We have no doubt that the reader who reached this point have already rolled up his sleeves and independently tries to find out what is the rank of Gerasimova in the problem of three bodies.

[2] Answer: because it is not necessary ....

4. The basics of the algebraic theory of Brahe–Descartes–Wotton.[3] Everywhere in this subsection the C-algebra A is a (differentially) finitely generated domain of integrity. First we show that the results of Section 2 are valid for arbitrary such C-algebras of rank 1.

4.1. Preliminary information.

Lemma 2. *If A has no free differential C-subalgebras, then A has an element a such that the localization A_a with respect to it is finitely generated as a commutatively associative C-algebra, i.e., A_a is an affine differential C-algebra and* $\mathrm{RANG}_C A \le \deg_C A = \deg_C A_a \ne \infty$.

Proof. Let $a_1, \dots, a_n$ be differential generators of A. Consider the chain of elements $a_i, a_i', \dots, a_i^{(q)}, \dots$. The hypothesis of the lemma implies that algebraically dependent elements must be among them. Let $P_i(x_0, x_1, \dots, x_{m_i})$ be a nonzero polynomial of the least degree satisfying the equality $P_i(a_i, a_i', \dots, a_i^{(m_i)}) = 0$. Differentiating this equality, we get the relation

$$0 = \left(\frac{\partial P_i}{\partial x_{m_i}} \cdot a_i^{(m_i+1)} + \sum_{j=0}^{m_i-1} \frac{\partial P_i}{\partial x_j} \cdot a_i^{(j+1)}\right)\Bigg|_{x_0=a_i, x_1=a_i', \dots, x_{m_i}=a_i^{(m_i)}}.$$

But in this case all $a_i^{(m_i+j)}$ $(j = 1, 2, \dots)$ lie in the localization A_{e_i} of the algebra A with respect to the element

$$e_i \overset{\text{def}}{=} \frac{\partial P_i}{\partial x_{m_i}}\Big|_{x_j=a_i^{(j)}\ (j=0,1,\dots,m_i)}.$$

Assume $a \overset{\text{def}}{=} e_1 \cdot e_2 \cdot \ldots \cdot e_n$. In this case the localization $A_a \overset{\text{def}}{=} A[a^{-1}]$ is a differential C-algebra generated by the elements a^{-1}, $a_i^{(j)}$, where $i = 1, \dots, n$, $j = 1, \dots, m_i$, as a commutatively associative C-algebra. This proves that A_a is an affine differential C-algebra. Other assertions of the lemma are evident now.

Corollary. *If A has no free differential C-subalgebras, then its analytic spectrum contains the main open set* $\{M \in \mathrm{Spec}_C A | a(M) \ne 0\}$.

Lemma 3. *If A contains a free differential C-subalgebra, then* $\mathrm{RANG}_C A = \infty$.

The proof immediately follows from the following fundamental assertion: *if A contains some free differential C-algebra, then A also contains a free differential C-subalgebra F_1 with one free generator x such that any maximal ideal in F_1 can be raised up to the maximal ideal in A* (see [7]). Therefore, $\mathrm{RANG}_C A \ge \mathrm{RANG}_C F_1 = \infty$. (In fact, consider the differential C-homomorphism $\varphi_m : F_1 \to C[[z]]$ such that $\varphi_m(x) \overset{\text{def}}{=} e^{\lambda_1 \cdot z} + \ldots + e^{\lambda_m \cdot z}$, where $\lambda_1, \dots, \lambda_m \in C$ are linearly independent over the subfield of rational numbers Q. In this case the differential C-subalgebra $\varphi_m(F_1)$ in $C[[z]]$ relative to the differentiation $\frac{d}{dz}$ contains all the exponentials $e^{\lambda_1 \cdot z}, \dots, e^{\lambda_m \cdot z}$ algebraically independent over C. Since any homomorphism $\varphi : F_1 \to C[[z]]$ is a Tailor homomorphism, then $\max_\varphi \deg \varphi(F_1) > 1, 2, \dots, m, \dots$, i.e., $\mathrm{RANG}_C F_1 = \infty$.) Lemma 3 is completely proved.

Theorem 1. *Any n-generated differential C-algebra B of rank 1 not containing nil-elements is a homeomorphic image of Gerasimova's differential C-algebra $G_{m,n}$ for sufficiently large natural number m. Moreover, all differential C-subalgebras in B have rank ≤ 1.*

Proof. According to the Ritt–Raudenbush theorem (see [8]), the C-algebra B is a finite subcartesian product of differential domains of integrity. More precisely, there exist differential ideals $I_1, \dots, I_s$ in B such that a) $I_1 \cap I_2 \cap \ldots \cap I_s = 0$, b) $A_i \overset{\text{def}}{=} B/I_i$ $(i = 1, \dots, s)$ are domains of integrity. By $b_1, \dots, b_n$ we denote the differential generators of B. Let $\varphi_i : B \to A_i$ be canonical (differential) epimorphisms. Since the rank of $\varphi_i(B)$ equals 1, then by Lemma 3 the algebra A_i cannot contain free differential C-subalgebras, and by Lemma 2 A_i is a differential C-subalgebra of some affine differential C-algebra and $\varphi_i(b_1), \dots, \varphi_i(b_n)$ are its differential generators. Therefore, the corollary from Proposition 3 implies that A_i is a homomorphic image of the C-algebra $G_{m_i,n}$ $(g_j \to \varphi_i(b_j),\ j = 1, \dots, n)$ and the following relations must hold on the generators $\{\varphi_i(b_j) | j = 1, \dots, n\}$:

a) $H_{m_i}(\varphi_i(b_j), \varphi_i(b_j')) = 0$ $(j = 1, 2, \dots, n)$;

b) $H_{m_i}(\varphi_i(b_j), \varphi_i(b_t)) = 0$ $(j > t; j, t = 1, 2, \dots, n)$.

Assuming $m \overset{\text{def}}{=} \max m_i$, from the latter we conclude (because of $I_1 \cap I_2 \cap \ldots \cap I_s = 0$) that the following relations hold for $b_1, \dots, b_n$:

a) $H_m(b_j, b_j') = 0$ $(j = 1, 2, \dots, n)$;

b) $H_m(b_j, b_t) = 0$ $(j > t; j, t = 1, 2, \dots, n)$,

[3]See also the same heading in [4].

and these relations imply that B is a homeomorphic image of the C-algebra $G_{m,n}$ ($g_j \to b_j$, $j = 1, \ldots, n$). This proves the first assertion of the theorem.

Repeating word for word the reasoning in the proof of the corollary of Proposition 3, we obtain that any (differential) finitely generated C-subalgebra in B has rank ≤ 1. But in this case for each differential C-subalgebra E in B and under the Tailor homomorphism $\widetilde{\psi}_M : E \to C[[z]]$ (M runs over $\mathrm{Spec}_{\mathbf{C}} E$) any power series $\widetilde{\psi}_M(e_1)$, $\widetilde{\psi}_M(e_2)$ (e_1, e_2 run over the whole E) must be connected by a nontrivial polynomial relation. Therefore, $\deg_C \widetilde{\psi}_M(E) \le 1$, and this means that $\mathrm{RANG}_C E \le 1$. The assertion of the theorem is completely proved.

Corollary. *A differential C-algebra B with differential generators $b_1, \ldots, b_q$ has rank ≤ 1 if and only if there exist natural numbers m, n such that*

a) $(H_m(b_j, b_j'))^n = 0$ $(j = 1, 2, \ldots, q)$;

b) $(H_m(b_i, b_j))^n = 0$ $(i > j; i, j = 1, 2, \ldots, q)$.

Proof. By Rad B we denote the set of all nil-elements in B. Since the algebra of power series $C[[z]]$ has no divisors of zero, then $\widetilde{\psi}_M(\mathrm{Rad}\ B) = 0$ for any $M \in \mathrm{Spec}_C B$, and this means that Rad B is a differential ideal in B and the ranks of the differential C-algebras B and $B/\mathrm{Rad}\ B$ coincide. Denote the canonical epimorphism $B \to B/\mathrm{Rad}\ B$ by φ.

If the rank of B equals one, then by Theorem 1 the factor algebra $B/\mathrm{Rad}\ B$ is a homomorphic image of some algebra $G_{m,q}$ ($g_i \to \bar{b}_i$ $(i = 1, \ldots q)$) and the following relations must hold on its generators $\bar{b}_1 \overset{\text{def}}{=} \varphi(b_1)$, $\ldots$, $\bar{b}_q \overset{\text{def}}{=} \varphi(b_q)$:

a) $H_m(\bar{b}_j, \bar{b}_j') = 0$ $(j = 1, 2, \ldots, q)$;

b) $H_m(\bar{b}_i, \bar{b}_j) = 0$ $(i > j; i, j = 1, 2, \ldots, q)$.

Therefore, all $H_m(b_j, b_j')$, $H_m(b_i, b_j)$ belong to Rad B, i.e., they are nil-elements. Taking n sufficiently large, we obtain the proof of the corollary in one direction.

If the differential generators $b_1, \ldots, b_q$ in the C-algebra B satisfy the relations indicated in the formulation of the corollary, then the generators $\bar{b}_1, \ldots, \bar{b}_q$ satisfy in $B/\mathrm{Rad}\ B$ the relations

a) $H_m(\bar{b}_j, \bar{b}_j') = 0$ $(j = 1, 2, \ldots, q)$;

b) $H_m(\bar{b}_i, \bar{b}_j) = 0$ $(i > j; i, j = 1, 2, \ldots, q)$,

i.e., $B/\mathrm{Rad}\ B$ is a homomorphic image of the differential C-algebra $G_{m,q}$ and

$$1 \ge \mathrm{RANG}_C(B/\mathrm{Rad}\ B) = \mathrm{RANG}_C(B),$$

which finishes the proof of the corollary.

4.2. Differential C-algebras of rank 1. By Γ_1 we denote the class of differential (countable-dimensional) C-algebras B such that all their (differentially) finitely generated C-subalgebras have rank ≤ 1. Obviously, this class is closed relative to the operations of determination of differential C-subalgebras and homomorphic images. A keen reader can easily notice that the differential C-algebras and C-subalgebras appearing in formulations of assertions belong to Γ_1. Recent results of Pogudin allow us to weaken the conditions posed on C-subalgebras of this class.

Theorem 2. *The class Γ_1 consists of differential C-algebras B such that any element w from B satisfies the equality $(H_m(w, w'))^n = 0$ $(m = m(w),\ n = n(w))$.*

The definition of the class Γ_1 and the corollary of Theorem 1 imply that the proof of Theorem 2 is reduced to justification of the following assertion.

Proposition 4. *A finitely generated differential C-algebra B has rank ≤ 1 if the rank of any its one-generated differential C-subalgebra does not exceed one.*

Prove this fact in three stages.

1. *The C-algebra B does not contain divisors of zero.* In this case the assertion of the proposition immediately follows from the fundamental result of Pogudin [9]: *if B has no free differential C-subalgebras, then the field of quotients of B coincides with the field of quotients $Q(A)$ of some differentially-one-generated in B C-subalgebra A.* (In fact, according to Lemma 3, the algebra B cannot contain one-generated free subalgebras isomorphic to F_1. Therefore, the differential generators $b_1, \ldots, b_q$ of the C-algebra B lie in $Q(A)$ and by Lemma 2 there exists an element a in A such that a) $b_1, \ldots, b_q \in A_a$; b) A_a is an affine differential C-algebra of rank ≤ 1, i.e., B is a (differentially) finitely generated C-subalgebra in A_a. Due to the corollary from Proposition 3, we get $\mathrm{RANG}_C B \le 1$.)

2. *The C-algebra B does not contain nil-elements.* In this case, as was indicated in the proof of Theorem 1, there exist differential ideals $I_1, \ldots, I_s$ in B such that a) $I_1 \cap I_2 \cap \ldots \cap I_s = 0$, b) $A_i \overset{\text{def}}{=} B/I_i$ $(i = 1, \ldots, s)$

are domains of integrity for which we have already proved the inequality $\mathrm{RANG}_C A_i \leq 1$ on the first stage. Therefore, A_i are homeomorphic images of some C-algebras $G_{m_i,q}$ of Gerasimova. In this case, taking $m = \max\{m_1, \ldots, m_s\}$, we conclude (due to $I_1 \cap I_2 \cap \ldots \cap I_s = 0$) that B is a homeomorphic image of $G_{m,q}$ and $\mathrm{RANG}_C B \leq 1$.

3. *The general case* Rad $B \neq 0$. As we have seen above, $\mathrm{RANG}_C B = \mathrm{RANG}_C(B/\mathrm{Rad}\ B)$. The factor-algebra $B/\mathrm{Rad}\ B$ has no nilpotent elements, therefore (as was proved on the second stage), its rank does not exceed one. Therefore, the rank of B does not exceed one too.

It was said (see [10]) about Tycho Brahe: "... Skill not pass. A man leaves, and it goes with him. Students, followers, and craft remain." So it goes, so to speak. The everlasting Cartesian question: "Should we take everything with us?"

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