# ROLLING SIMPLEXES AND THEIR COMMENSURABILITY, IV.

## A Farewell to Arms! [1]

**Gold Fish, Master Key**

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*Abstract.* The text is both an algebraic lesson and a Cartesian master class (a gift of Gods, one should say) for metaphysicians, metaphysicists and their followers.

The authors consider their duty to express deep gratitude towards Igor Rostislavovich Shafarevich for his abiding interest in the results of our studies and support of our work.

*"We catch in flight all dreams of realists,*
*In diff-equations we are the best.*
*We are simple algebraists, not physicists,*
*Light good old Mind is our quest!"*
*From the L.V.M. manifest.*

We start with one intuitionistic, purely algebraic trick, which enables us to resolve several kinds of differential equations with respect to the highest order derivative.

**Lemma on affinity of intermediate subalgebra.** Assume that a finitely generated commutative associative integral domain $A$ is an algebra with the identity element over an algebraically closed field $k$ ($\mathrm{char} k = 0, 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, \dots$), and its quotient field $Q(A)$ is of transcendency degree 1 over $k$. Further, if algebras $A$ and $B$ in a sequence of $k$-subalgebras $A \subseteq C \subseteq B \subset Q(A)$ are finitely generated, than the same is true of the subalgebra $C$.

**Proof.** As soon as $B$ is finitely generated, $C$ is a countably dimensional $k$-algebra. We choose elements $\{e_i | i = 1, 2, \dots\}$ in $C$ as to complement the basis of the $k$-algebra $A$ to a basis of $C$. Assign $C_0 \stackrel{\text{def}}{=} A$, $C_{i+1} \stackrel{\text{def}}{=} C_i[e_{i+1}]$, $C_\infty \stackrel{\text{def}}{=} B$ and denote by $\bar{C}_0, \bar{C}_1, \dots, \bar{C}_i, \dots, \bar{C}_\infty$ the integral closures ("normalizations") of these finitely generated subalgebras in $Q(A)$. It is well known that each $k$-subalgebra of that type is finitely generated. Moreover, in the course of the proof (see [1]) it is established that $\bar{C}_i$ is finitely generated as a module over $C_i$, and consequently Noetherian. Let us show that the ascending chains of $k$-subalgebras $\{\bar{C}_i\}$, $\{C_i\}$ $(i = 1, 2, 3, \dots)$ eventually stabilize.

**Proposition 1.** If in a chain of integral domains $F \subseteq G \subset Q(F)$ the $k$-subalgebras $F$ and $G$ are finitely generated, $F = \bar{F}$ (i.e. $F$ is integrally closed), and $\deg_k Q(F) = 1$, then the natural mapping $\nu : \mathrm{Spec}_k G \to \mathrm{Spec}_k F$ satisfies the following conditions:

(a) $\nu$ is injective,

(b) if $\nu$ is surjective, then $G$ coincides with $F$,

(c) $(\mathrm{Spec}_k F) \setminus \nu(\mathrm{Spec}_k G)$ is a finite set.

That is an exact translation of the statement of Corollary 2 of Theorem 2 in section 2 of part 2 (see the monograph [1]).

From the property (b) we conclude that if $\bar{C}_i \neq \bar{C}_{i+1}$, then there exists a maximal ideal $M \in \mathrm{Spec}_k \bar{C}_i$ that cannot be raised to an ideal in $\mathrm{Spec}_k \bar{C}_{i+1}$, thus it follows from the property (a) that $M \cap \bar{C}_0 \in \mathrm{Spec}_k \bar{C}_0$ cannot be raised to an ideal in $\mathrm{Spec}_k \bar{C}_\infty$. But by virtue of the property (c) there are finite many maximal ideals in $\bar{C}_0$ that cannot be raised to ideals in $\mathrm{Spec}_k \bar{C}_\infty$. Consequetly, there are finitely many places where the inclusions are strict in the ascending chain of "normalizations" $\bar{C}_0 \subseteq \bar{C}_1 \subseteq \dots \bar{C}_m \subseteq \dots$, i.e. $\bar{C}_N = \bar{C}_{N+i}$ for a sufficiently large number $N \in N$ $(i = 1, 2, \dots)$ and $C_N \subseteq C = \bigcup_m C_m \subseteq \bar{C}_N$. But as noted above $\bar{C}_N$ is Noetherian as a module over $C_N$. Thus its

[1] **Differential-algebraic curves do not exist.**

$C_N$-submodule $C$ is finitely generated and has to coincide with $C_{N+q}$ for a particular $q \in N$, which proves the proposition.

The following statements are easily deduced from the proposition above.

**Theorem 1.** Any finitely generated differential $k$-algebra without zero divisors and of transcendence degree 1 is finitely generated as a commutative associative algebra, in particular this differential $k$-algebra is finitely presented.

**Corollary.** The spectrum of all the maximal ideals $\mathrm{Spec}_{\mathbb{C}}A$ of an arbitrary finitely generated differential commutative associative $\mathbb{C}$-algebra $A$ without zero divisors and of transcendence degree 1 is analytic, i.e. for any $\mathbb{C}$-homomorphism $\psi_M : A \to \mathbb{C}$ ($M \in \mathrm{Spec}_{\mathbb{C}}A$) under the Taylor homomorphism $\widetilde{\psi}_M : A \to \mathbb{C}[[z]]$ all the series $\widetilde{\psi}_M(a) \overset{\mathrm{def}}{=} \sum\limits_{m=0}^{\infty} \psi_M(a^{(m)})\frac{z^m}{m!}$ converge in some neighborhood of zero.

**Theorem 2.** Let $X$ be an affine irreducible algebraic curve over an algebraically closed field $k$ and $k[X]$ be its algebra of regular functions. Then any $k$-subalgebra in $k[X]$ is generated by a finite subset of its elements.

**Corollary.** Let $K$ be a field of transcendence degree 1 over its algebraically closed subfield $k$, $\mathrm{Der}_k K$ an algebra Lie of all $k$-derevations of the $K$. Then for any $a_1$, ..., $a_m \in K$, $D_1$, ..., $D_n \in \mathrm{Der}_k K$ the least commutative assotiative $k$-subalgebra $A$ in $K$ containing $a_1$, ..., $a_m$, for which $D_i(A) \subseteq A$ $(i = 1, 2, \dots, n)$, is finitely generated.

We should like to illustrate these results and their proofs by particular examples.

**1. Differential Picard algebras** (see [2]). [2] Let $P$ be a differential $\mathbb{C}$-algebra determined by generators $x_1, \dots, x_n$ and $n$ defining relations: $x_i' = f_i(x_1, \dots, x_n)$ $(i = 1, 2, \dots, n)$, where $f_i$ are elements of the polinomial algebra $\mathbb{C}[x_1, \dots, x_n]$. It is evident that the algebra $P$ arbitrary chosen contains no zero divisors and can be realised by the derivation $D \overset{\mathrm{def}}{=} \sum\limits_{i=1}^{n} f_i \frac{\partial}{\partial x_i}$ on $\mathbb{C}[x_1, \dots, x_n]$.

The spectrum of this differential algebra coincides with the affine space $\mathbb{C}^n$. The estimate $|((D^m \times f)|_{x=M})/m!| \leq n^m a^{m+1}$ holds for all coefficients of the Taylor series $\widetilde{\psi}_M(f) \overset{\mathrm{def}}{=} \sum\limits_{m=0}^{\infty} (D^m \times f)|_{x_i=\alpha_i,\dots,x_n=\alpha_n} \cdot \frac{z^m}{m!}$ $(f \in \mathbb{C}[x_1, \dots, x_n], M \overset{\mathrm{def}}{=} (\alpha_1, \dots, \alpha_n))$, where $a$ is the largest absolute value of the functions $f$, $f_1, \dots, f_n$ and all their partial derivatives at the point $M$. Consequently, all the power series $\widetilde{\psi}_M(x_1), \dots, \widetilde{\psi}_M(x_n)$ converge in some neighborhood of zero. The equality $\widetilde{\psi}_M(f) = f(\widetilde{\psi}_M(x_1), \dots, \widetilde{\psi}_M(x_n))$ shows us that for any $f \in \mathbb{C}[x_1, \dots, x_n]$ the series $\widetilde{\psi}(f)$ converges at the same neighborhood. Since any finitely generated commutative associative $\mathbb{C}$-algebra $A$ with a fixed derivation $D \in \mathrm{Der}_{\mathbb{C}} A$ is a homomorphic image of the Picard algebra $P$ for a suitable choice of $n$ and $f_1, \dots, f_n$, the spectrum $\mathrm{Spec}_{\mathbb{C}} A$ is also analitic for any $D \in \mathrm{Der}_{\mathbb{C}} A$. That proves the corollary of Theorem 1.

**2. "Rational" differential-algebraic parametrisations of flat affine irreducible algebraic curves.** Denote by $X_H$ a flat affine irreducible algebraic curve given by an equation $H(x, y) = 0$ $(H(x, y) \in k[x, y])$. Let $k[X_H]$ be $k$-algebra of regular functions of $X_H$ over algebraically closed field $k$ (char $k = 0, 1, 2, \dots$. In sections 2.1-3, 3 we suppose that all differential $k$-algebras ("parametresations") defined by differential algebraic relations are in a natural way to contain $k[X_H]$ as a $k$-subalgebra. Of coure, it implies that the irreducible polynomial $H(x, y)$ does have additional properties. For once we point out the following exact and evident conditions

а) $\frac{\partial H}{\partial y} \not\equiv 0$ in the sections 2.1, 2.3,

б) $x \cdot \frac{\partial H}{\partial x} + y \cdot \frac{\partial H}{\partial y} \notin k \cdot H(x, y)$ in section 2.2,

в) $(\frac{\partial H}{\partial x})^2 + (\frac{\partial H}{\partial y})^2 \not\equiv 0$ in section 3,

which are necessary and sufficient for embedding $k[X_H]$ in corresponding $k$-algebras given below.

**2.1. Singly generated differential-algebraic curves (proof of Theorem 2).** We determine a differential $k$-algebra with the identity element $W_H$ by two generators $\omega$, $\omega_1$ and two defining relations $H(\omega, \omega_1) = 0$, $\omega' = \omega_1$, where $H(\omega, \omega_1)$ is an irreducible polynomial in $k[\omega, \omega_1]$, for which $\frac{\partial H}{\partial \omega_1} \not\equiv 0$. Unfortunately, it is currently unknown whether this commutative associative algebra contains zero

[2] **Proof of the corollary of Theorem 1.**

divisors. To get rid of such kind of virtual elements (when $\frac{\partial H}{\partial \omega_1} \not\equiv 0$) we consider in $W_H$ the differential ideal $I_H \stackrel{\text{def}}{=} \{a \in W_H | \left(\frac{\partial H}{\partial \omega_1}\right)^m \cdot a = 0, m = m(a)\}$ and localize $W_H$ by the element $d \stackrel{\text{def}}{=} \frac{\partial H}{\partial \omega_1} \in W_H$. Then the kernel of the canonical homomorphism of differential algebras $\nu : W_H \to (W_H)_d \stackrel{\text{def}}{=} \{\frac{b}{d^k} | b \in W_H\}$ coincides with the ideal $I_H$. Let $\overline{W}_H \stackrel{\text{def}}{=} \nu(W_H)$, $\bar{\omega} \stackrel{\text{def}}{=} \nu(\omega)$, $\bar{d} \stackrel{\text{def}}{=} \nu(d)$, $\bar{\omega}_1 \stackrel{\text{def}}{=} \nu(\omega_1)$, $k[X_H] \stackrel{\text{def}}{=} k[\bar{\omega}, \bar{\omega}_1]$. Now from the equality $\bar{\omega}' = \bar{\omega}_1$ we conclude that $\overline{W}_H$ is differentially generated by one element $\bar{\omega}$, and the equality $0 = H' = \frac{\partial H}{\partial \omega}\omega' + \frac{\partial H}{\partial \omega_1}\omega''$ leads to the fact that under $\nu : W_H \to (W_H)_d$ all elements of $\nu(W_H) = \overline{W}_H$ lie in the commutative associative subalgebra generated by three elements $\omega$, $\omega_1$, $d^{-1} = (\omega_1 \cdot \frac{\partial H}{\partial \omega_1})^{-1}$.

Now we are in a position to realize $\overline{W}_H$ as a differential $k$-subalgebra in the field $k(X_H)$ with respect to the derivation $D \stackrel{\text{def}}{=} \omega_1(\frac{\partial}{\partial \omega} - (\frac{\partial H}{\partial \omega} / \frac{\partial H}{\partial \omega_1})\frac{\partial}{\partial \omega_1})$, where $X_H$ is the flat irreducible affine algebraic curve given by equation $H(\omega, \omega_1) = 0$, Consequently, we get a chain of $k$-algebras $k[X_H] \subseteq \overline{W}_H \subseteq (\overline{W}_H)_{\bar{d}} \subseteq k(X_H)$, which satisfies all the conditions of the lemma on affinity of intermediate subalgebra. It is evident that *any one generated differential subalgebra of an arbitrary integral domain (of transcendence degree 1) is a homomorphic image of* $\overline{W}_H$ *for an appropriate choice of* $H(\omega, \omega_1)$ ($\frac{\partial H}{\partial \omega_1} \not\equiv 0$). That proves Theorem 1 on affinity of differential-algebraic curves.

(Especially note that the above reasoning is true for all algebraically closed fields of positive characteristic $p$, because the equalities $\frac{\partial H}{\partial w_1} \equiv 0$, $\frac{\partial H}{\partial \omega} \equiv 0$ imply another relation $H(\omega, \omega_1) = (F(\omega, \omega_1))^p$, which contradicts the irreducibility of $H$.)

We conclude the section by one not complicated (maybe whatnot, but memorable) version of Theorem 1.

**Proposition 2.** If in a differential domain of integrity $F$ over an algebraically closed field $k$ elements $f$ and $f'$ satisfy to a nonzero polynomial relation $H(f, f') = 0$ ($H(x, y) \in k[x, y]$, $H \not\equiv 0$), then there exists a natural number $N$, for which $f^{(N)} = G(f, f', f", \ldots, f^{(N-1)})$, where $G(x_1, x_2, \ldots, x_N) \in k[x_1, x_2, \ldots, x_N]$.

**Corollary.** If in a real interval $(a, b)$ an infinitely differentiable complex valued function $f(t)$ is a solution of a differential equation $H(f, f') = 0$, where $H(x, y) \in \mathbf{C}[x, y]$, is a nonzero irreducible polynomial, then $f^{(N)}(t) = G(f(t), f'(t), f"(t), \ldots, f^{(N-1)}(t))$, where $G(x_1, x_2, \ldots, x_N) \in \mathbf{C}[x_1, x_2, \ldots, x_N]$, for a sufficiently large natural number $N$.

**2.2. Kepler parametrizations of flat curves.** A differential $k$-algebra with the identity element $G_H$ is given by generators $x$, $y$ and two differential defining relations $H(x, y) = 0$, $xy' - x'y = \sigma$, where $H$ is an irreducible polynomial in $k[x, y]$, for which $x \cdot \frac{\partial H}{\partial x} + y \cdot \frac{\partial H}{\partial y} \notin k \cdot H(x, y)$ [3], and $0 \neq \sigma \in k$ (for example, $\sigma = \hbar / m_e$). Solving the system of equations $0 = H' = \frac{\partial H}{\partial x} \cdot x' + \frac{\partial H}{\partial y} \cdot y'$, $-yx' + xy' = \sigma$ with respect to $x', y'$, we get $\mathcal{L}(x, y) \cdot \begin{pmatrix} x' \\ y' \end{pmatrix} = \sigma \begin{pmatrix} -\partial H / \partial y \\ \partial H / \partial x \end{pmatrix}$, where $\mathcal{L} \stackrel{\text{def}}{=} \frac{\partial H}{\partial x} \cdot x + \frac{\partial H}{\partial y} \cdot y$. If the irreducible affine curve $X_H$ (given by the equation $H(x, y) = 0$) is smooth, the ideal generated by $\frac{\partial H}{\partial x}$, $\frac{\partial H}{\partial y}$ in $k[X_H]$ has to coincide with the algebra $G_H$. Thus $a\frac{\partial H}{\partial x} + b\frac{\partial H}{\partial y} = 1$ for appropriate $a$, $b$ from $k[X_H]$. Consequently, $\mathcal{L} \cdot (-ax' + by') = \sigma$, i.e. the element $\mathcal{L}$ is invertible in $G_H$. It follows immediately that $G_H$ (as a commutative associative $k$-algebra)

(a) is generated by its three elements: $x$, $y$, $\mathcal{L}^{-1}$;

(b) can be included into the field $k(X_H)$ and contains no zero divisors;

(c) is realized as a differential subalgebra in $k(X_H)$ with respect to the derivation $D_H \stackrel{\text{def}}{=} \sigma \cdot \mathcal{L}^{-1}(-\frac{\partial H}{\partial y}\frac{\partial}{\partial x} + \frac{\partial H}{\partial x}\frac{\partial}{\partial y})$.

In general it is possible that there exist zero divisors in $G_H$. Let $I$ be an arbitrary differential ideal in $G_H$, for which $G_H / I$ contains no zero divisors. If we assume that $I$ intersects by more than just the zero element with the subalgebra $k[X_H]$ generated by $x$, $y$ in $G_H$, then the algebra $k[X_H]/(I \cap k[X_H])$

[3] *If* $H = H_0 + H_1 + \cdots + H_m$, *where* $H_0$, $H_1$, ..., $H_m$, *are homogeneous components of the polynomial* $H(x, y)$ *then* $x \cdot \frac{\partial H}{\partial x} + y \cdot \frac{\partial H}{\partial y} = 0 \cdot H_0 + 1 \cdot H_1 + \cdots + m \cdot H_m$ *and in the case, when* char $k = 0$, *the inclusion* $x \cdot \frac{\partial H}{\partial x} + y \cdot \frac{\partial H}{\partial y} \in k \cdot H(x, y)$ *is true for homogeneous* $H(x, y)$ *only. But an arbirary irreducible homogeneous polynomial has degree 1. Hence,* $H(x, y) = H_1$ *and the equation* $H(x, y) = 0$ *defines a strait line, which contains the origin.*

be zero-dimensional, and the integral domain $G_H/I$ should coincide with $k\cdot 1$, but it contradicts the statement $xy'-x'y=\sigma\neq 0$. If $I\cap k[X_H]=0$, then the element $\mathcal{L}\in k[X_H]$ is not equal to zero in the integrity domain $G_H/I$ and after localization by $\mathcal{L}$ we get $(G_H/I)_{\mathcal{L}}=(G_H)_{\mathcal{L}}/I_{\mathcal{L}}$. Consequently, the ideal $I$ has to coincide with the ideal $I(H)\stackrel{\text{def}}{=}\{a\in G_H|\mathcal{L}^m\cdot a=0 \text{ in } G_H, m=m(a)\}$. It proves that there exists the only integral domain $\bar{G}_H$, given by generators $x$, $y$ and two differential defining relations $H(x,y)=0$, $xy'-x'y=\sigma$ ($\sigma\in k,\sigma\neq 0$), for which the following statements are true:

(a) $\bar{G}_H$ is imbeded into $k(X_H)$ with respect to the derivation $D_H\stackrel{\text{def}}{=}\mathcal{L}^{-1}\cdot\sigma(-\frac{\partial H}{\partial y}\frac{\partial}{\partial x}+\frac{\partial H}{\partial x}\frac{\partial}{\partial y})$;

(b) under this embedding $k[X_H]\subseteq\bar{G}_H\subseteq(\bar{G}_H)_{\mathcal{L}}\subset k(X_H)$ and as a commutative associative $k$-algebra the localization $(\bar{G}_H)_{\mathcal{L}}$ is generated by three elements $x,y,\mathcal{L}^{-1}$;

(c) $\bar{G}_H$ is a simple differential $k$-algebra and the signature derivation does not vanish at any point of the spectrum $\mathrm{Spec}_k\bar{G}_H$.

Hereby (look also at Lemma on affinity of intermediate subalgebra), $X_{\bar{G}_H}=\mathrm{Spec}_k\bar{G}_H$ is a smooth affine irreducible algebraic curve and $\bar{G}_H$ contains $k[X_H^\nu]$, where $X_H^\nu$ is a normalization of the curve $X_H$. So we see that the Kepler observer $\bar{G}_H$ excludes from his consideration all the non-linear branches of $X_H^\nu$, and slightly moving off the origin, can notice all the linear ones.

This observer surveying the curve $X_H$ can try to act more radically: run up along the line $x=0$ from the origin to the gallery.

**2.3 Puiseux parametrizations.** Let us consider a differential $k$-algebra with the identity element $P_H$, given by generators $x$, $y$ and two defining relations $H(x,y)=0$, $x'=c$ ($c\in k$, $c\neq 0$), where $H(x,y)$ is an irreducible polynomial, for which $\frac{\partial H}{\partial y}\not\equiv 0$. The arguments presented in the previous two examples ensure that there exits a unique (possibly zero) differential ideal (equal to $I\stackrel{\text{def}}{=}\{a\in P_H|\left(\frac{\partial H}{\partial y}\right)^m\cdot a=0, m=m(a)\}$) such as the quotient algebra with respect to that ideal contains no zero divisors. Denote this quotient algebra throw $\bar{P}_H$. The equality $0=H'=\frac{\partial H}{\partial x}c+\frac{\partial H}{\partial y}y'$ shows us that the localization $\bar{P}_H$ with respect to the element $\frac{\partial H}{\partial y}$ is generated as a commutative associative $k$-algebra by its three elements $x,y,\left(\frac{\partial H}{\partial y}\right)^{-1}$, and $\bar{P}_H$ is realized as a differential subalgebra in the field $k(X_H)$ with respect to the derivation $D=D(H)=c\left(\frac{\partial}{\partial x}-\left(\frac{\partial H}{\partial x}/\frac{\partial H}{\partial y}\right)\frac{\partial}{\partial y}\right)$. Then $k[X_H]\subset\bar{P}_H\subseteq(\bar{P}_H)_{\frac{\partial H}{\partial y}}\subset k(X_H)$, and in the view of uniqueness of the ideal $I$ the differential $k$-algebra $\bar{P}_H$ does not vanish at any point of $\mathrm{Spec}_k\bar{P}_H$ and in the same way as in the previous example $X_{\bar{P}_H}\stackrel{\text{def}}{=}\mathrm{Spec}_k\bar{P}_H$ is a smooth affine irreducible algebraic curve, for which $k[X_{\bar{P}_H}]=\bar{P}_H$ contains $k[X_H^\nu]$, where $X_H^\nu$ - is a normalization of the flat curve $X_H^\nu$. Wherein $\bar{P}_H$ excludes from its consideration those branches of the curve $X_H^\nu$, for which their projections of the tangents on the plane $Oxy$ are parallel to the line $x=0$. (Non-linear branches are excluded, too).

**2.4. Common case:** $\left(D=\frac{P}{Q}\left(\frac{\partial H}{\partial y}\frac{\partial}{\partial x}-\frac{\partial H}{\partial x}\frac{\partial}{\partial y}\right)\right):k(X_H)\to k(X_H)$ is treated similarly 2.1 - 2.3.

**3. Fermat parametrizations (natural parameter).** Let us consider a differential $k$-algebra with the identity element $F_H$ determined by generators $x$, $y$ and two defining relations $H(x,y)=0$, $(x')^2+(y')^2=c^2$ ($\mathrm{char}k\neq 2$, $0\neq c\in k$), where $H(x,y)$ is an irreducible polynomial in $k[x,y]$, for which $\Delta\stackrel{\text{def}}{=}(\frac{\partial H}{\partial x})^2+(\frac{\partial H}{\partial y})^2\not\equiv 0$. [4] It is obvious that signature derivation does not vanish at any point of the spectrum $\mathrm{Spec}_kF_H$ of the $k$-algebra $F_H$, thus if we show that any homomorphic image $\bar{F}_H$ of the algebra $F_H$ without zero divisors is of transcendence degree 1 over $k$, then $\bar{F}_H$ will turn up to be a simple finitely generated differential $k$-algebra with an analytic spectrum. Let $\phi:F_H\to\bar{F}_H$ be the corresponding epimorphism, $\bar{x}\stackrel{\text{def}}{=}\phi(x),\bar{y}\stackrel{\text{def}}{=}\phi(y)$, $k[X_H]$ the algebra of regular functions of the flat affine curve $X_H$ given by the equation $H(x,y)=0$. It is clear [5] that $k[X_H]$ is isomorphic to

[4] *If* $\Delta=0$, *then* $0=(\frac{\partial H}{\partial x}+(-1)^{1/2}\cdot\frac{\partial H}{\partial y})\cdot(\frac{\partial H}{\partial x}-(-1)^{1/2}\cdot\frac{\partial H}{\partial y})=\frac{\partial H}{\partial t_1}\cdot\frac{\partial H}{\partial t_2}$, *where* $t_1\stackrel{\text{def}}{=}(1/2)\cdot(x+(-1)^{1/2}\cdot y)$, $t_2\stackrel{\text{def}}{=}(1/2)\cdot(x-(-1)^{1/2}\cdot y)$. *But then, if* char $k=0$, *(due to irreducibility H(x,y) over an algebraically closed field) either* $H=\alpha\cdot t_1+\beta$, *or* $H=\alpha\cdot t_2+\beta$. *In that case the condition* $(\frac{\partial H}{\partial x})^2+(\frac{\partial H}{\partial y})^2\not\equiv 0$ *excludes from our consideration two bundles of parallel direct lines given by the following equations* $x+y\cdot(-1)^{1/2}=\delta$, $x-y\cdot(-1)^{1/2}=\delta$, *for which* $x'^2+y'^2=(x'+y'\cdot(-1)^{1/2})\cdot(x'-y'\cdot(-1)^{1/2})=0$.

[5] *The equality* $H(x,y)=0$ *implies the relation* $0=H=x'\cdot\frac{\partial H}{\partial x}+y'\cdot\frac{\partial H}{\partial y}$, *in particular,* $x'^2\cdot(\frac{\partial H}{\partial x})^2=y'^2\cdot(\frac{\partial H}{\partial y})^2$ *and*

the $k$-subalgebra, generated by $x$, $y$ in $F_H$ and $k[X_H] \cap \mathrm{Ker}\phi = 0$ (otherwise the zero-dimensional subalgebra $\phi(k[X_H])$ would generate $\bar{F}_H$ and $\bar{x}', \bar{y}'$ should be equal to zero, but this contradicts with the equality $(x')^2 + (y')^2 = c^2 \neq 0$). Equalities $\frac{\partial H}{\partial x} \equiv 0$, $\frac{\partial H}{\partial y} \equiv 0$ are possible if $\mathrm{char}k = p > 0$, but due to the fact that $H(x, y)$ is irreducible, either $\frac{\partial H}{\partial x} \not\equiv 0$, or $\frac{\partial H}{\partial y} \not\equiv 0$.

We consider the case when $\frac{\partial H}{\partial y} \not\equiv 0$. Then $d \stackrel{\text{def}}{=} \phi\left(\frac{\partial H}{\partial y}\right)$ is a nonzero element in the integral domain $\bar{F}_H$ and from the equalities $0 = \phi(H') = \phi(\frac{\partial H}{\partial x})\bar{x}' + \phi(\frac{\partial H}{\partial y})\bar{y}'$, $(\bar{x}')^2 + (\bar{y}')^2 = c^2$ we get that $\bar{y}' = -(\phi(\frac{\partial H}{\partial x})/d)\bar{x}'$, $(\bar{x}')^2(1 + (\phi(\frac{\partial H}{\partial x})/d)^2) = c^2$ in the quotient field $Q(\bar{F}_H)$ of the algebra $\bar{F}_H$. The last relation implies that

(a) the $k$-subalgebra $E$ generated by $\bar{x}$, $\bar{y}$, $\bar{x}'$, $\bar{y}'$ is contained in the "quadratic extension" of the field $\phi(k(X_H))$;

(b) $\bar{x}^{(i)}, \bar{y}^{(i)} \in Q(E) (i = 2, 3, \dots)$.

These properties prove that the integral domain $\bar{F}_H$ is contained in $Q(E)$ and $\deg_k \bar{F}_H = \deg_k Q(E) = 1$. By Theorem 1, the commutative associative $k$-algebra $\bar{F}_H$ is finitely generated and finitely presented as a differential $k$-algebra. As the signature derivation $'$ does not vanish at any point of $X_{\bar{F}_H} \stackrel{\text{def}}{=} \mathrm{Spec}_k \bar{F}_H$, $\bar{F}_H$ is an integrally closed $k$-algebra and $\bar{F}_H$ contains $k[X_H^\nu]$, where $X_H^\nu$ is the normalization of the curve $X_H$.

**4. Nonaffine differential-algebraic surfaces exist.** Let a differential $\mathbb{C}$-algebra $E$ (with the identity element) be given by generators $x$, $y$ and two defining relations $x' = 1$, $x^2y' + y - x = 0$. Assume that $\bar{x}(z) \stackrel{\text{def}}{=} z$, $\bar{y}(z) \stackrel{\text{def}}{=} \sum_{m=0}^{\infty} (-1)^m m! \cdot z^{(m+1)}$ and generate a differential $\mathbb{C}$-subalgebra $\bar{E}$ by $\bar{x}$, $\bar{y}$ in power series $\mathbb{C}[[z]]$ with respect to the derivation $\frac{d}{dz}$. Direct verification shows that $\frac{d\bar{x}}{dz} = 1$, $\bar{x}^2\frac{d\bar{y}}{dz} + \bar{y} - \bar{x} = 0$ in $\mathbb{C}[[z]]$. Thus, the integral domain $\bar{E}$ is a homomorphic image of $E$ at $\phi : E \to \bar{E}$ ($\phi(x) = \bar{x}$, $\phi(y) = \bar{y}$) and we consistently get

(a) $\mathrm{Ker}\phi = \{a \in E | x^{2m} \cdot a = 0 \ (m = m(a))\}$,

(b) for the maximal ideal $M \in \mathrm{Spec}_{\mathbb{C}}\bar{E}$, which is the intersection of $\bar{E}$ with the unique maximal ideal of $\mathbb{C}[[z]]$, under the Taylor homomorphism $\widetilde{\psi}_M : \bar{E} \to \mathbb{C}[[z]]$ we get $\widetilde{\psi}_M(\bar{x}) = \bar{x}$, $\widetilde{\psi}_M(\bar{y}) = \bar{y}$, i.e. $\mathrm{Spec}_{\mathbb{C}}\bar{E}$ is not analytic at the point $M$;

(c) $\bar{x}$, $\bar{y}$ are algebraically independent over $\mathbb{C}$ (otherwise $\bar{E}$ would coincide with some Puiseux parametrization $\bar{P}_H$ ($H(x, y) \in \mathbb{C}[x, y]$) and $\mathrm{Spec}_{\mathbb{C}}\bar{E}$ would be analytic);

(d) $\bar{E}$ can be realized in the field of rational functions $\mathbb{C}(x, y)$ as a differential $\mathbb{C}$-subalgebra with respect to the derivation $D \stackrel{\text{def}}{=} \frac{\partial}{\partial x} + \frac{x-y}{x^2}\frac{\partial}{\partial y}$.

In this way the differential integral domain $\bar{E}$ has transcendence degree equal to 2 over $\mathbb{C}$, its spectrum of maximal ideals is not analytic and, consequently, (as a commutative associative $\mathbb{C}$-algebra) $\bar{E}$ cannot be finitely generated.

We leave it as an exercise for the reader to verify two more properties of the $\mathbb{C}$-algebra $\bar{E}$:

(e) $\mathbb{C}[x, y] \subset \bar{E} \subset \mathbb{C}[x, y, x^{-1}] \subset \mathbb{C}(x, y)$;

(f) $\bar{E}$ is a simple differential $\mathbb{C}$-algebra.

**5. Proof of Theorem 2.** An arbitrary $k$-subalgebra $C$ of the finitely generated algebra $k[X]$ is countably dimensional. If $C$ contains the identity element of $k[X]$, then we can choose a basis $\{e_i | i = 0, 1, \dots\}$ in $C$, where $e_0 \stackrel{\text{def}}{=} 1$. Let $C_0 \stackrel{\text{def}}{=} k \cdot e_0$, $C_{i+1} \stackrel{\text{def}}{=} C_i[e_{i+1}]$ $(i = 0, 1, 2, \dots)$. As soon as the field $k$ is algebraically closed, the $k$-algebra $C_1$ is isomorphic to the polynomial algebra $k[e_1]$. Let us consider an ascending chain of quotient fields $Q(C_i)$ of $k$-algebras $C_i$. By virtue of the fact that $k[X]$ is finitely generated and $\deg_k k(X) = 1$, the field $k(X)$ is a finite extension of the field $Q(C_1)$ and $\dim_{Q(C_1)} Q(C_i) \leq \dim_{Q(C_1)} Q(C_{i+1}) \leq \dim_{Q(C_1)} k(X)$. Consequently, the ascending chain of fields $Q(C_i)$ $(i = 0, 1, 2, \dots)$ is stabilized for a sufficiently large integer $N$, i.e. $Q(C_N) = Q(C_{N+i})$ $(i = 1, 2, \dots)$. Let $A \stackrel{\text{def}}{=} C_N$, then $Q(A) = Q(C) \subseteq k(X)$. Embedding $A \subset k[X]$ defines a natural

*taking into consideration the equality $x'^2 + y'^2 = c^2$ we have that $x'^2 \cdot \Delta = c^2 \cdot (\frac{\partial H}{\partial x})^2$, $y'^2 \cdot \Delta = c^2 \cdot (\frac{\partial H}{\partial y})^2$. As soon as $\Delta \not\equiv 0$ and $\Delta = (\frac{\partial H}{\partial x} + (-1)^{1/2} \cdot \frac{\partial H}{\partial y}) \cdot (\frac{\partial H}{\partial x} - (-1)^{1/2} \cdot \frac{\partial H}{\partial y})$, the irreducible polynomial $H(x, y)$ does not divide the $\Delta$, i,e. $\Delta \neq 0$ in $k[X_H]$ and $(\bar{x}')^2$, $(\bar{y}')^2 \in k[X_H]_\Delta \subset k(X_H)$.*

regular map $\nu : X \to X_A \stackrel{\text{def}}{=} \mathrm{Spec}_k A$. As soon as $\deg_k k(X) = 1$, the set $X_A \backslash \nu(X)$ is finite and $X_A$ contains a finite number of singularities. Thus, in the $k$-algebra $A$ we can choose an element $d$, for which the following statements are true

(a) the localization $A_d \stackrel{\text{def}}{=} A[d^{-1}] \subset Q(A)$ of the algebra $A$ with respect to the element $d$ is an integrally closed $k$-algebra;

(b) the localization $(k[X])_d$ is composed by all the algebraic elements over $A_d$ and any ideal from $\mathrm{Spec}_k A_d$ can be raised to an ideal from $\mathrm{Spec}_k (k[X])_d$, in particular, to an ideal from $\mathrm{Spec}_k (C_{N+i})_d$. By Proposition 1, where $F = A_d, G = (C_{N+i})_d$, we conclude that $A_d = (C_{N+i})_d = C_d$ and we have a chain of subalgebras $A \stackrel{\text{def}}{=} C_N \subseteq C \subseteq B \stackrel{\text{def}}{=} A_d = (C_{N+i})_d \subseteq Q(A)$ satisfying all the conditions of Lemma on affinity of intermediate subalgebra. It completes the proof of Theorem 2 in the case when the $k$-subalgebra $C$ contains the identity element of $k[X]$.

Otherwise, we consider the $k$-subalgebra $C_{\text{id}} \stackrel{\text{def}}{=} k \cdot 1 \oplus C$, which, as proved above, is generated by a finite subset of its elements: $e_i = \lambda_i \cdot 1 \oplus c_i$ $(i = 1, \ldots, m,\ m = m(C),\ c_i \in C,\ \lambda_i \in k)$. But then $c_1, \ldots, c_m \in C$ generate $C$. Theorem 2 is completely proved.

## Bibliography


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